Solutions:

1.1

a) Yes.
b) No.
c) Yes.
d) No.
e) Yes.
f) No.
g) No.
h) Yes.
i) No.

1.2

a) **A)**.
b) **B)**.
c) **C)**.

1.3

a) No.
b) Yes.
c) No.
d) Yes.
e) No.

1.4 **A)**.

1.5
$k \neq 2$.

1.6
B), **C)**.

1.7 **B)**.

1.8 **B)**.

1.9 $\mathbf{w} = (a, b, c) = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$.

1.10

a) $a \neq 0 \wedge b \neq 3$.
b) $a \neq 0$.

1.11 Use the definition of linearly independent vectors. Show that

$$\alpha(\mathbf{u} - \mathbf{v}) + \beta(\mathbf{u} - 2\mathbf{v} + \mathbf{w}) = 0_{\mathscr{E}} \Rightarrow \alpha = \beta = 0$$

1.12 $\alpha = -2$.

1.13 $\alpha = -2$, $\beta = -1$.

1.14

a) Yes.
b) No.
c) Yes.
d) Yes.

1.15 **D)**.

1.16

i) True.
ii) True.
iii) True.
iv) False.
v) True.
vi) False.
vii) False.
viii) True.

1.17

a) **B)**.
b) **C)**.

1.18 **A**.

1.19 **D)**.

1.20

a) No.
b) Yes.
c) No.
d) No.
e) Yes.

1.21

a) Show that the basis consist of two independent vectors. As $\dim \mathbb{R}_1[x] = 2$, a basis consists of two independent vectors.
b) **B)**.

1.22 **D)**.

1.23 **B)**.

1.24

a) $(x,y,z) = \left(\frac{x-2y-4z}{3}\right)(1,1,-1) + \left(\frac{2x-y+z}{3}\right)(2,1,0)\left(\frac{-x+2y+z}{3}\right)(2,3,-1)$
b) Yes, the vectors are linearly independent and generators, so they are a basis of $\mathbb{R}^3$.

1.25 $\dim \mathscr{S} = 3$, basis for $\mathscr{S}$: $\{(2,1,1),(1,2,5),(1,-1,4)\}$.

1.26 We have three vectors of $\mathbb{R}^3$. To show that they form a basis of $\mathbb{R}^3$ we just need to show that they are linearly independent. Since $\dim \mathbb{R}^3 = 3$, then three vectors linearly independent are also generators of $\mathbb{R}^3$. We obtain:

$$\alpha \mathbf{u}_1 + \beta \mathbf{u}_2 + \gamma \mathbf{u}_3 = \mathbf{0}_{\mathbb{R}^3} \Leftrightarrow$$
$$\begin{cases} \alpha + \beta + \gamma = 0 \\ -\beta + \gamma = 0 \\ \alpha + 2\beta + 3\gamma = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

The three vectors are linearly independent, thus form a basis of $\mathbb{R}^3$.

1.27

a) $b + 1 \neq a$.
b) $(1,2,0) = \frac{3}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 - \frac{1}{2}\mathbf{u}_3$.

1.28

a) $b(x) = x$.
b) $[-2,2,-7]$.

1.29

b) $2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3 = \frac{3}{2}\mathbf{f}_1 - \frac{1}{2}\mathbf{f}_2 - 2\mathbf{f}_3$.
c) Think of how to add e_3.

1.30

a) Yes.
b) No.
c) No.
d) Yes.
e) No.
f) Yes.
g) No.
h) Yes.
i) Yes.
j) Yes.

1.31 We need to show that $(0,0)$ is a solution of equation $x_1 + 8x_2 = 0$. It is easy verified. The other condition to prove is that $\alpha(x_1,x_2) + \beta(y_1,y_2)$ is also a solution of the equation, where $\alpha, \beta \in \mathbb{R}$. If (x_1,x_2) and (y_1,y_2) are solutions of the equation, then $x_1 = -8x_2$ and $y_1 = -8y_2$. We obtain:

$$\alpha(x_1,x_2) + \beta(y_1,y_2) = \alpha(-8x_2,x_2) + \beta(-8y_2,y_2) = (-8\alpha x_2, \alpha x_2) + (-8\beta y_2, \beta y_2) =$$
$$= (-8(\alpha x_2 + \beta y_2), \alpha x_2 + \beta y_2) \Leftrightarrow -8(\alpha x_2 + \beta y_2) + -8(\alpha x_2 + \beta y_2) = 0$$

This the conditions are satisfied, so the set of all pairs $(x_1,x_2) \in \mathbb{R}^2$ that are solutions of the equation $x_1 + 8x_2 = 0$ is a subspace of $\mathbb{R}^2$.

1.32 Other.

1.33

a) $\mathbb{R}^2$.
b) $\{(x,y,z) \in \mathbb{R}^3 :\ y = 0\}$.
c) $\{(x,y,z) \in \mathbb{R}^3 :\ 3x - 3y - 2z = 0\}$.
d) $\{(x,y,z,t) \in \mathbb{R}^4 :\ t = 0\}$.

1.34

a) $(2,1)$
b) $((1,0,-1),(0,1,-1))$.
c) $((2,1,0),(1,0,1))$.
d) $((-2,1,1,\frac{1}{2}),\ (1,1,0,0))$.

1.35

a) As $\mathscr{E}$ is a vector space of dimension 3, we only need to show that the three vectors $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ are generators of $\mathscr{E}$. This will prove that they are also linearly independent. We write the condition satisfied by the vectors as $\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3=\mathbf{x}_4$. This shows that $\mathbf{x}_4$ is generated by the three vectors. Thus $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ are generators of $\mathscr{E}$.
b) We can follow the same reasoning as before by writing the condition as $\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_4=\mathbf{x}_3$.

1.36

a) **C)**.
b) **A)**.
c) **D)**

1.37

a) **C)**.
b) **A)**.

1.38 **B)**.

1.39 **B)**.

1.40

a) Show that the vectors $\mathbf{f}_i$, $i=1,2,3$, are linearly independent. As they are 3 and $\dim\mathscr{F}=3$, then they are a basis of $\mathscr{F}$.
b) $(-1,-\frac{1}{3},\frac{4}{3})$.
c) $\alpha=-\frac{3}{7}b$, $\beta=\frac{1}{3b}$, com $b\neq 0$, with $(a,b)=(-\frac{7}{3},b)$ coordinates of $\mathbf{u}$ in the basis of $\mathscr{F}$.
d) Impossible. Why?
e)
  i)
  ii) (e_1-e_2,e_3).
  iii) $\mathscr{F}\cap\mathscr{G}=\{x_1e_1-x_1e_2-2x_1e_3,\ x_1\in\mathbb{R}\}$. $\mathscr{F}+\mathscr{G}=\{(a+b+c)e_1+(b-c)e_2+(-a+d)e_3,\ af_2+bf_1\in\mathscr{F},\ c(e_1-e_2)+de_3\in\mathscr{G}\}$. Basis of $\mathscr{F}\cap\mathscr{G}$ is $(e_1-e_2-2e_3)$. Basis of $\mathscr{F}+\mathscr{G}$ is (e_1-e_2,e_3,f_2,f_1).

1.41 A generic vector $\mathbf{w}$ of $\mathbb{R}^2$ is written as (w_1,w_2). The vector $\mathbf{w}$ may be written as a linear combination of one vector in $\mathscr{F}$ and one vector in $\mathscr{G}$. We obtain $\mathbf{w}=\alpha(x,0)+\beta(0,y)$, where $\alpha,\beta,\ x,\ y\in\mathbb{R}$. Thus $w_1=\alpha x$ and $w_2=\beta y$. We conclude that any vector if $\mathbb{R}^2$ is written as a linear combination of subspaces $\mathscr{F},\mathscr{G}$ and as so $\mathscr{E}=\mathscr{F}+\mathscr{G}$.

1.42 Let $a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n+\ldots$, where $a_0,a_1,a_2,\ldots,a_n,\ldots\in\mathbb{R}$, be a generic vector of $\mathbb{R}[x]$. This vector can be written in the form $(a_0+a_2x^2+a_4x^4\ldots+a_nx^n+\ldots)+(a_1x+a_3x^3+a_5x^5\ldots)$, so it is a linear combination of the subspaces $\mathscr{F}$ and $\mathscr{G}$. Thus,
$\mathbb{R}[x]=\mathscr{F}+\mathscr{G}$.

1.43

a) $\mathscr{F}+\mathscr{G}=((1,1,-1),(0,1,0),(0,0,1))$.
b) $(x,y,z)=x(1,1,-1)+(y-x)(0,1,0)+(x+z)(0,0,1)$.

1.44

a) $(x,y,z) = z(1,-1,1) + (y+z)(2,1,0),\ \wedge\ x - 2y - 3z = 0$. $\dim \mathscr{F} = 2$.
b) $\mathscr{F} \cup \mathscr{G} = \{(x,y,z) \in \mathbb{R}^3 :\ x = 2y - z \wedge x = 2y + 3z\}$. It is not a subspace of $\mathbb{R}^3$. Why?

1.45

a) $(x,y,z,t) = z(1,-1,1,0) - t(2,1,0,-1),\ \wedge\ x = z - 2t,\ y = -z - t$. $\dim \mathscr{F} = 2$.
b) $\mathscr{F} \cap \mathscr{G} = \langle (1,-1,1,0) \rangle$.

1.46 $\mathscr{F} \cap \mathscr{G} = ((2,1,0,0))$.

1.47

a) Yes for both.
b) $\mathscr{F} \cap \mathscr{G} = ((0,1,0))$.

2.1

a) If $c \neq 0$ then the application is not a linear transformation, otherwise it is a linear transformation.
b) No.
c) Yes.
d) No.
e) Yes.

2.2 $T(1,0,0,0) = (1,1,1)$; $T(0,1,0,0) = (0,1,0)$; $T(0,0,1,0) = (0,0,0)$; $T(0,0,0,1) = (0,0,0)$.

2.3

a) $M(T, \mathbf{b}, \mathbf{b}') = \begin{bmatrix} 0 & -2 \\ 2 & 2 \\ 0 & -2 \end{bmatrix}$
b) $\ker\ T = \langle (0,0) \rangle$, $\dim(\ker\ T) = 0$. $\mathrm{im}\ T = \langle (1,2,0),(1,0,0) \rangle$, $\dim(\mathrm{im}\ T) = 2$.

2.4 **C)**.

2.5 **B)**

2.6

a) **C)**.
b) **D)**.

2.7

a) **B)**.
b) **A)**.

2.8

a) **D)**.
b) **C)**.
c) **C)**.
d) **A)**.

2.9

a) **B)**.
b) **A)**.
c) **A)**.
d) **B)**.

2.10

a) **C)**.
b) **D)**.

2.11

a) $\ker\ T = \langle(2,-2,1)\rangle$, $\dim\ker\ T = 1$. $\mathrm{im}\ T = \langle(1,-1),(1,0)\rangle$, $\dim\mathrm{im}\ T = 2$.
b) $T(1,0,1) = 1\cdot T(1,0,0) + 0\cdot T(0,1,0) + 1\cdot T(0,0,1) = (1,-1)+(0,2) = (1,1)$.

2.12

a) $T(1,1) = (-1,2)$, $T(0,1) = (-3,1)$.
b) $M(T,\mathbf{b},\mathbf{b}') = \begin{bmatrix} -5 & -5 \\ -3 & -4 \end{bmatrix}$

2.13

a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.14 $T(x,y,z) = (x+z, x+y, x+2y)$

2.15

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

b) $\mathbf{B} = \begin{bmatrix} \frac{1}{3} & 1 \end{bmatrix}$

c) $\mathbf{C} = \mathbf{BA} = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{bmatrix}$

d) $(T_2 \mathbf{o} T_1)(x,y,z) = 3\left(-\frac{1}{3}x - \frac{2}{3}y - \frac{2}{3}z\right)$

2.16

a) $(0,0,0,0)$.
b) No. The kernel is not reduced to $\mathbf{0} \in \mathbb{R}^4$!

2.17 Show using the domains and ranges of the two linear transformations.

2.18 **B)**.

2.19 **C)**.

2.20

a) **B)**.
b) **C)**.
c) **A)**.

2.21

a) **C)**.
b) **D)**.
c) **D)**.

2.22

a) **B)**.
b) **A)**.
c) **A)**.
d) **B)**.

2.23

a) **B)**.
b) **C)**.
c) **A)**.

2.24 **D)**.

2.25

a) $(2,3,-1,0)$.

b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

c) The basis of ker f is $\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right)$.

d) $\begin{bmatrix} 2 & 1 & -1 & 2 \\ -3 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

2.26

a) **D)**.
b) **D)**.
c) **B)**.

2.27

a) **D)**.
b) **B)**.
c) **D)**.

2.28

a) **C)**.
b) **D)**.

3.1 **A)**.

3.2 **B)**.

3.3 **D)**.

3.4 **B**).

3.5 **C**).

3.6 **A**).

3.7 **A**).

3.8 **A**).

3.9 **B)**

3.10

a) **B**).
b) **C**).
c) **C**).
d) **C**).

3.11 **D)**

3.12 **D**).

3.13 **C**).

3.14

a) **B**).
b) **A**).

3.15 **D**).

3.16 **D**).

3.17 **D**).

3.18 **B**).

3.19 **B**).

3.20 **C**).

3.21 **C**).

3.22 **D**).

3.23 **B**).

3.24 **D**).

3.25

a) **A**).
b) **D**).

3.26 **C**).

3.27

(a) To show that $\mathbf{AA}^T$ is symmetric, we need to verify the equality $\left(\mathbf{AA}^T\right)^T = \mathbf{AA}^T$. By the properties of the transpose, we obtain:

$$\left(\mathbf{AA}^T\right)^T = \left(\mathbf{A}^T\right)^T \mathbf{A}^T = \mathbf{AA}^T$$

To show that $\mathbf{A}^T\mathbf{A}$ is symmetric, we must show that $\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\mathbf{A}$. We use again the properties of the transpose. Thus:

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T \left(\mathbf{A}^T\right)^T = \mathbf{A}^T\mathbf{A}$$

(b) If $\mathbf{A}$ is symmetric then $\mathbf{A} = \mathbf{A}^T$. If $\mathbf{A}$ is invertible then there exists $\mathbf{A}^{-1}$ satisfying:

$$\mathbf{AA}^{-1} = \mathbf{I}_n \equiv \mathbf{A}^T\mathbf{A}^{-1} = \mathbf{I}_n$$

and

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n \equiv \mathbf{A}^{-1}\mathbf{A}^T = \mathbf{I}_n$$

Thus, $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}^T$.

(c) Since $\mathbf{A}$ is invertible then $\mathbf{A}^T$ is also invertible. Since the product of invertible matrices is invertible then $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are also invertible.

3.28 **D**).

3.29

a) **D**).
b) **A**).

3.30 **C**).

3.31 **D**).

3.32 **D**).

3.33 **C**).

3.34 **D**).

3.35 **B**).

3.36 **C**).

3.37

a) **A**).
b) **A**).

3.38 **B**).

3.39 **C)**

3.40 **B)**

3.41

a) **D**).
b) **A**).

3.42 **D**).

3.43 **D**).

3.44 **B**).

3.45 **C**).

3.46 **D**).

3.47 **A**).

3.48

a) **D**).
b) **D**).

3.49

a) **B**).
b) **C**).
c) **A**).

4.1 No. The second and third columns have been switched in the matrix of system S_2, comparatively to the matrix of system S_1. In order for S_1 to be equivalent to S_2, that is, in order to these systems to have the same solutions, that switch should have been followed by a switch in the second and third variables, y and z, in the second system S_2. As this is not the case, the solutions of the two systems are not equal, thus the systems are not equivalent.

4.2

a) Consistent-dependent system (has only one solution): $\mathbf{X} = [5, -\frac{5}{2}, \frac{5}{2}]^T$.
b) Consistent-independent system (has multiple solutions): $\mathbf{X} = [\frac{2}{5} + \frac{3}{5}t, \frac{1}{2} - t, \frac{1}{10} - \frac{3}{5}t, t]^T$, $t \in \mathbb{R}$.
c) Consistent-dependent system (has only one solution): $\mathbf{X} = [769, -767, 385, -95, 13, -1, 1]^T$.

4.3

a) $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

b) $\mathbf{B}^{-1} = \begin{bmatrix} \frac{13}{6} & \frac{1}{6} & \frac{3}{2} \\ -1 & 0 & -1 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$

c) $\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{8} & -\frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{8} & -\frac{5}{8} & \frac{13}{8} \end{bmatrix}$

d) $\mathbf{D}^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & -\frac{2}{5} & 1 \\ \frac{3}{8} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{2} \\ -\frac{9}{40} & \frac{1}{20} & \frac{7}{40} & \frac{1}{2} \\ \frac{1}{20} & \frac{1}{10} & -\frac{3}{20} & 0 \end{bmatrix}$

4.4 **C**).

4.5 **B**).

4.6 **B**).

4.7 **B**).

4.8

a) **D**).
b) **C**).
c) **B**).

4.9 **C**).

4.10 **D**).

4.11

a) **A**).
b) **B**)

4.12

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 \\ 0 & 0 & \frac{1}{\delta} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix}, \qquad \mathbf{B}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\gamma} \\ 0 & 0 & \frac{1}{\delta} & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 \\ \frac{1}{\alpha} & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & 0 \\ -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 & 0 \\ \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} & 0 \\ -\frac{1}{\alpha^4} & \frac{1}{\alpha^3} & -\frac{1}{\alpha^2} & \frac{1}{\alpha} \end{bmatrix}, \mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & 0 \\ -\frac{1}{\alpha\beta} & \frac{1}{\beta} & 0 & 0 \\ \frac{1}{\alpha\beta\delta} & -\frac{1}{\beta\delta} & \frac{1}{\delta} & 0 \\ -\frac{1}{\alpha\beta\delta\gamma} & \frac{1}{\beta\delta\gamma} & -\frac{1}{\delta\gamma} & \frac{1}{\gamma} \end{bmatrix}$$

4.13 **D**).

4.14

a) **C).**
b) **B).**
c) **B).**

4.15 **D**).

4.16

a) For $\alpha = 5 \wedge \beta = -1/5$ the system is consistent-independent. For $\alpha\beta \neq 1$ the system is consistent-dependent. For $\alpha\beta = 1 \wedge \alpha \neq 5$, the system is inconsistent.

b) For $a+9=0$ the system is inconsistent. For $a+9 \neq 0$ the system is consistent-dependent.

c) For $\alpha \neq -6/7$ the system is consistent-dependent. For $\alpha = -6/7 \wedge \beta = 35/3$ the system is consistent-independent. For $\alpha = -6/7 \wedge \beta \neq 35/3$ the system is inconsistent.

4.17 **C**).

4.18 **B**)

4.19 **A**).

4.20 **B**).

4.21

a) **D**).
b) **B**).

4.22

a) **D)**.
b) **A)**.

4.23

a) $\alpha = -1 \vee \alpha = \frac{4}{3}$.
b) We choose $\alpha = 1$. The solution in this case is $[\frac{39}{5} \ \ -\frac{19}{5} \ \ -\frac{11}{5}]^T$.

4.24

a) **B)**.
b) **A)**.
c) **B)**.

4.25

a) **C)**.
b) **D)**.
c) **A)**.

4.26

a) **B)**.
b) **D)**.
c) **D)**.

4.27

a) **D)**.
b) **D)**.
c) **A)**.

4.28

a) **A)**.
b) **A)**.

4.29 **B)**.

4.30

a) **C)**.
b) **D)**.
c) **A)**.
d) **C)**.

4.31

a) **C)**.
b) **A)**.
c) **B)**.

5.1

a) $|\mathbf{B}^2| = |\mathbf{B}||\mathbf{B}| = |\mathbf{B}|^2 = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}^2 = \left(3\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}\right)^2 = 0^2 = 0$. (properties used: the determinant of a product is the product of the determinants; multilinearity)

b) $|\mathbf{C}| = 1(-1)(-1) = -1$ (properties used: the determinant of a triangular matrix is the product of the elements in the diagonal).

c)
$$|\mathbf{D}| = \begin{vmatrix} 42 & 5 & 1 \\ 84 & 0 & 0 \\ 63 & 5 & 2 \end{vmatrix} = 5\begin{vmatrix} 42 & 1 & 1 \\ 84 & 0 & 0 \\ 63 & 1 & 2 \end{vmatrix} = 21 \cdot 5 \begin{vmatrix} 2 & 1 & 1 \\ 4 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 105 \begin{vmatrix} 2 & 1 & 0 \\ 4 & 0 & 0 \\ 3 & 1 & 1 \end{vmatrix} =$$
$$= -105 \begin{vmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} = -105 \cdot 4 = -420$$

(properties: multilinearity; determinant of a triangular matrix).

5.2

a) det $\mathbf{A} = -1$
b) det $\mathbf{B} = 0$
c) det $\mathbf{C} = 5$.

5.3 We choose the third column for Laplace's expansion. We obtain:

$$\det \mathbf{A} = 1(-1)^{1+3} \begin{vmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} = \left[-\frac{2}{3}(-1)^{1+1} \cdot \left(-\frac{1}{4}\right) + \frac{1}{3}(-1)^{1+2} \cdot \frac{1}{4} \right] = \frac{1}{12}$$

5.4 **A**).

5.5

a) $1, \forall \theta \in [0, 2\pi]$.
b) 0.
c) $12 - 31c + c^3 - 6c^2$.

5.6 **C**).

5.7 **D**).

5.8 **D**).

5.9 **A**).

5.10 $x = -9$.

5.11 **D**).

5.12 **A**).

5.13 **D**).

5.14 **C)**

5.15 **B)**

5.16 **D)**

5.17 **D))**

5.18 **D)**

5.19 **B)**

5.20 Note that the third column is a linear combination of the first and second columns. Thus the determinant is zero. Prove it step by step using the properties.

5.21 $\det\ \mathbf{AB} = 198$ and $\det\ \mathbf{A} \cdot \det\ \mathbf{B} = 198$

5.22 $\det\ \mathbf{A} = -29 = \frac{1}{\det\ \mathbf{A^{-1}}}$, thus $\det\ \mathbf{A}^{-1} = -\frac{1}{29}$.

5.23

b) $\det\ \mathbf{A}^T = \det\ \mathbf{A} = (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$, thus $\det\ \mathbf{A}^T = \det\ \mathbf{A}$.

5.24 **B)**.

5.25

a) **D)**.
b) **C)**.
c) **D)**.

5.26

a) $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} \\ -\frac{4}{5} & \frac{1}{5} \end{bmatrix}$.

b) $\mathbf{B}^{-1} = \begin{bmatrix} \frac{92}{148} & \frac{3}{148} & \frac{10}{148} \\ -\frac{56}{148} & \frac{3}{148} & \frac{10}{148} \\ \frac{4}{148} & \frac{13}{148} & -\frac{6}{148} \end{bmatrix}$.

c) $\mathbf{C}^{-1} = \begin{bmatrix} \frac{14}{42} & -\frac{14}{42} & \frac{14}{42} \\ -\frac{40}{42} & \frac{34}{42} & -\frac{16}{42} \\ \frac{38}{42} & -\frac{26}{42} & \frac{11}{42} \end{bmatrix}$.

5.27

a) **B)**.
b) **D)**.

5.28

5.29 **A)**.

5.30 **C)**.

5.31 **A)**.

5.32 **C)**.

5.33 **B)**.

5.34 **D)**.

5.35 **B)**.

5.36

a) **C**.
b) **D**.
c) **C)**.
d) **A)**.

5.37

a) **C)**.
b) **D)**.
c) **A)**.

5.38

a) **A)**.
b) **D)**.

5.39 **D)**.

5.40 **A)**.

5.41 **D)**.

5.42 **B)**.

5.43 **A)**.

5.44

a) **B)**.
b) **D)**.

5.45 **D)**.

5.46

a) **D)**.
b) **A)**.

5.47

a) **A)**.
b) **D)**.

5.48

a) **C)**.
b) **C)**.
c) **A)**.

5.49 Eigenvalue $\lambda_1 = 1$, algebraic multiplicity 2, geometric multiplicity 1. The proper subspace is generated by the vector $(1,0,-4)$.
Eigenvalue $\lambda_2 = 3$, algebraic multiplicity 1, geometric multiplicity 1. The proper subspace is generated by the vector $(1,2,4)$.

5.50 Eigenvalue $\lambda = 0$, with algebraic multiplicity 4 geometric multiplicity 1. Proper subspace generated by the vector $(0,0,1,0)$.

5.51 Eigenvalue $\lambda_1 = \frac{1}{2}$, with algebraic multiplicity 1 and geometric multiplicity 1. The eigenspace associated to eigenvalue λ_1 is generated by $(1,-\frac{1}{4},0,0)$.
Eigenvalue $\lambda_2 = 3$, with algebraic multiplicity 2 and geometric multiplicity 1. The eigenspace associated to eigenvalue λ_2 is generated by $(7,0,0,1)$.
Eigenvalue $\lambda_3 = 1$, with algebraic multiplicity 1 and geometric multiplicity 1. The eigenspace associated to eigenvalue λ_3 is generated by the vector $(1,0,0,0)$.

5.52 **C)**.

5.53

a) **C**).
b) **B**).
c) **B**).

5.54

a) **B**).
b) **A**).

5.55

a) **D**).
b) **A**).

5.56

a) **C**).
b) **B**).
c) **C**).

5.57 **C**).

5.58

a) **B**).
b) **D**).
c) **B**).
c) **D**).

5.59 **A**).

5.60 **A**).

6.1

a) $\sqrt{5}$.
b) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$.
c) -1.
d) $(2,2,1)$.
e) $\simeq 108.43$.

6.2 **A**).

6.3 **A**).

6.4 **A**).

6.5 **A**).

6.6
To show that the equalities hold, let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$. The, use the definitions of norm, scalar product and cross product, given in the beginning of this chapter. Solve each part of the equalities separately and show that they are equal.

6.7

a) **C**).
b) **C**).

6.8

a) **D**).
b) **C**).

6.9 **A)**

6.10 **A**).

6.11 **B**).

6.12 **A**).

6.13

a) **B**).
b) **D**).

6.14

a) **B**).
b) **A**).
c) **C**).

6.15

a) **B**).
b) **A**).

6.16 **D**).

6.17 **D**).

6.18 **A**).

6.19 **D**).

6.20

a) **B**).
b) **B**).
c) **D**).

6.21 **A**).

6.22

a) Show that the two straight lines are parallel to the same vector **u**.
b) $3x-6y+2z=-5$.
c) $\mathrm{d}(\mathbf{r_1},\mathbf{r_2})=1$.

6.23

a) **D**).
b) **B**).

6.24

a) $d(P,\alpha)=\frac{\sqrt{2}}{3}$.
b) $d(\alpha,\beta)=\frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$.

6.25

a) $\{(x,y,z) \in \mathbb{R}^3 :\ 2x+4y+4z=11\}$.
b) $P_2 \equiv (0, \frac{7}{4}, 1)$.

6.26 $(x,y,z) = (0,-1,0) + \lambda(-1,-1,-1),\ \lambda \in \mathbb{R}$.

6.27 $a = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}$.

6.28 $\beta\ :\ \begin{cases} x = \sqrt{2} + \lambda - \mu \\ y = -\lambda \\ z = \lambda - \mu \end{cases}, \quad \lambda, \mu \in \mathbb{R}.$

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